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CUBIC CURVES IN RECIPROCAL TRIANGULAR SITUATION

BY JOHN FREDERICK MESSICK

Introduction. The purpose of this paper is to find two cubic curves in what we shall call reciprocal triangular situation; that is, two cubics so related to each other that a single infinity of triangles with vertices on one may have their sides tangent to, or as we say, on the other, and also that the sides of a singly infinite system of triangles with vertices on the second may be on the first.

1. The Curve of Order and Class Three.

We shall start with a cuspidal cubic and write it in the form

$$x_1 x_3^2 = x_2^3$$

to which any cuspidal cubic may be reduced.*

We see at once from this equation that, taking the triangle with sides x_1, x_2, x_3, and vertices 1, 2, 3 as reference triangle, there is a cusp at the vertex 1, a flex at 3, x_3 is the cusp tangent, and x_1 the flex tangent. But we wish to see this from the parametric form of the cubic. This is

$$x_1 : x_2 : x_3 = \lambda^3 : \lambda : 1,$$

since the coördinates of any point on the cubic may be expressed as λ^3, λ, 1, where λ is a variable parameter.

Substituting these in the equation of any line $(\xi x) \equiv \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 = 0$, we get

$$\xi_1 \lambda^3 + \xi_2 \lambda + \xi_3 = 0.$$

The λ^2 term is lacking; therefore the sum of the roots of this equation, say λ_1, λ_2, λ_3, vanishes; and the parameters of the three points on the cubic in which it is cut by a straight line are connected, therefore, by the relation

$$\lambda_1 + \lambda_2 + \lambda_3 = 0;$$

or as we shall say, this is the involution for points of the cubic lying on a straight line.

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* Salmon, *Higher Plane Curves*, page 180.

The triple points of the involution are given when the λ's become equal, or by

$$3\lambda = 0,$$

the three roots of which are ∞, ∞, 0.

Assuming only that the cubic is rational, we know that the possible cases in which a line may cut a cubic in three coincident points, that is, at triple points, are the two nodal tangents, the cusp tangent, and the flex tangent. The possibilities then for singular points are node, cusp, and flex. Any line through a node or cusp cuts there in two coincident points. In the case of a node, when the curve is expressed parametrically, these two coincident points are given by distinct values of the parameter. But from the roots of our equation the two values of the parameter are the same, namely ∞. We infer, therefore, that if the equation giving the triple points has two equal roots, the cubic has a cusp given by this double value of the parameter and a flex given by the other root. Thus, as is obvious from the equation for a rational cubic, the triple points of the involution give the flexes, the neutral pair give the node. For a cuspidal cubic the neutral pair are equal, and two triple points are also equal. Thus, the condition for a cuspidal cubic is that the equation for the triple points of the involution shall have equal roots. Our cubic, therefore, has a cusp for the double root ∞, that is, at the point 1, 0, 0, and a flex for the root 0 at 0, 0, 1.

From these considerations we shall be able to determine readily the character of a cubic appearing in a more complicated form.

2. The Cubics ϕ and f. If a conic be passed through three given points on the cubic, it will cut the cubic in three other points. The six points in which the conic cuts the cubic are obtained by substituting the parameters λ^3, λ, 1 for x_1, x_2, x_3 respectively in the equation of the conic $(\xi x)^2 = 0$. This gives

$$\xi_1^2\lambda^6 + 2\xi_1\xi_2\lambda^4 + 2\xi_1\xi_3\lambda^3 + \xi_2^2\lambda^2 + 2\xi_2\xi_3\lambda + \xi_3^2 = 0.$$

Here the term in λ^5 is wanting; the sum of the roots of the equation, say λ_1, λ_2, λ_3, λ_4, λ_5, λ_6, is therefore equal to zero. We have then as a symmetrical relation of the λ's independent of the ξ's

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 0,$$

which we may say is the involution along the cubic for points on a conic.

Since we suppose three of the points fixed, the sum of their parameters therefore being a constant, we may say that the involution for the other three is

$$\lambda_1 + \lambda_2 + \lambda_3 = \kappa,$$

which we shall call an I_1^3, meaning by this symbol an involution of three things in which two must be given to determine the third. By I_2^3 is meant an involution in which, if one is given, the other two are determined.

The conic expressed in terms of symmetric functions of the parameters of points which it cuts out is

$$x_1^2 + S_2 x_1 x_2 - S_3 x_1 x_3 + S_4 x_2^2 - S_5 x_2 x_3 + S_6 x_3^2 = 0.$$

By breaking up the six points into two sets of three, say σ and σ', the involution already found is

$$\sigma_1 + \sigma_1' = 0\,;$$

or substituting $-\kappa$ for one of the sets which we suppose fixed, σ' say, we have it in the form given,

$$\sigma_1 = \lambda_1 + \lambda_2 + \lambda_3 = \kappa.$$

Now by putting the conic through a point a not on the cubic, we get $a_1^2 + a_1 a_2(\sigma_2 + \sigma_1\sigma_1' + \sigma_2') - a_1 a_2(\sigma_3 + \sigma_2\sigma_1' + \sigma_1\sigma_2' + \sigma_3') + a_2^2(\sigma_3\sigma_1' + \sigma_2\sigma_2' + \sigma_1\sigma_3') - a_2 a_3(\sigma_3\sigma_2' + \sigma_2\sigma_3') + a_3^2\sigma_3\sigma_3' = 0$. Since σ' is fixed this may be put in the form

$$A_0\sigma_3 + A_1\sigma_2 + A_2\sigma_1 + A_3 = 0,$$

where the A's are functions of the known quantities a and σ'. And since σ' is a constant, this may be written, if we exclude the special case when $A_0 = 0$,* as

$$\lambda_1\lambda_2\lambda_3 - a(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) - a\beta\kappa = 0,$$

a second relation among the varying λ's, also an I_1^3.

Through four given points — three on and one off the cubic — a whole pencil of conics may be passed, each cutting the cubic in three other points; hence, a single infinity of triads, or sets of three points, arise. If these be joined, a system of lines is given, the envelope of which is the curve we wish to obtain.

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* This means that a value of the parameter is ∞; that is, the point is taken at the cusp, which case we shall except.

This system of triangles is such that, if one side is given, the other two are determined. If we take a line, therefore, through two points of the cubic, and subject it to the two involutions

$$\lambda_1 + \lambda_2 + \lambda_3 = \kappa,$$

$$\lambda_1\lambda_2\lambda_3 - a(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) - a\beta\kappa = 0,$$

the combination of which is an I_2^3, the locus of the line will be the required curve.

The points of the varying triad, as we have supposed, are given by the parameters λ_1, λ_2, λ_3; and the equation of a line through two of these, say those given by λ_1 and λ_2, is, writing it in determinant form,

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \lambda_1^3 & \lambda_1 & 1 \\ \lambda_2^3 & \lambda_2 & 1 \end{vmatrix} = 0,$$

which becomes on reduction

$$x_1 - (\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)x_2 + \lambda_1\lambda_2(\lambda_1 + \lambda_2)x_3 = 0.$$

And now if we put the conditions given by the two involutions (the two I_1^3's) on this line, as the conic varies the line will describe the curve in question.

Put λ for λ_3 in the involutions, and eliminate λ_1 and λ_2 from the three equations. The result is

$$(\lambda - a)x_1 - \{\lambda(\lambda - \kappa)^2 - a\beta\kappa + a\kappa\lambda - a\kappa^2\}x_2$$
$$+ a\{\lambda(\lambda - \kappa)^2 - \beta\kappa(\lambda - \kappa)\}x_3 = 0.$$

This for a fixed λ is a line of the required curve; and for varying λ it gives the curve itself. And since it is of the third degree in λ, it is a cubic. Also, since three lines may be drawn from a point tangent to the curve, it is of the third class.

We have now the two cubics, the original in points,

$$\phi: \begin{cases} x_1 = \lambda^3 \\ x_2 = \lambda \\ x_3 = 1, \end{cases}$$

and the other, also written in parametric form, in lines,

$$f: \quad \left\{ \begin{aligned} \xi_1 &= \lambda - a \\ \xi_2 &= - \{ \lambda(\lambda - \kappa)^2 + a\kappa(\lambda - \beta - \kappa) \} \\ \xi_3 &= a \{ \lambda(\lambda - \kappa)^2 - \beta\kappa(\lambda - \kappa) \}, \end{aligned} \right.$$

so related that triangles may have their vertices on ϕ and their sides on f; or as we shall say, they lie in triangular situation.

To get the relation of the parameters of three points on ϕ, such as λ_3, the parameter of which gives the line through λ_1 and λ_2, when the lines of f given by these parameters meet in a point, substitute the ξ's of f in the equation of a point $(x\xi) = 0$. This gives, writing it in powers of λ,

$$(x_2 - ax_3)\lambda^3 - 2\kappa(x_2 - ax_3)\lambda^2 - (x_1 - \kappa^2 x_2 - a\kappa x_2 + a\kappa^2 x_3 - a\beta\kappa x_3)\lambda$$
$$+ ax_1 - a\beta\kappa x_2 - a\kappa^2 x_2 - a\beta\kappa^2 x_3 = 0.$$

Comparing this with the equation of symmetric functions, we get as a symmetrical relation of the λ's independent of the x's

$$\lambda_1 + \lambda_2 + \lambda_3 = 2\kappa.$$

The triple lines of this involution are given by

$$3\lambda = 2\kappa,$$

the three roots of which are ∞, ∞, $\frac{2\kappa}{3}$. Here again, since the cubic is rational, and the equation giving the triple lines has a double root, it is a curve of class three with a flex, that is, a cuspidal cubic. As this is the dual of the other case, here the double root ∞ gives the flex tangent, and the single root $\frac{2\kappa}{3}$ gives the cusp tangent of f.

On the substitution of ∞ in the equation of ϕ the flex tangent is seen to be

$$x_2 - ax_3 = 0.$$

Since x_1 is missing, this is a line through the reference point 1, 0, 0. The flex line of f, therefore, passes through the cusp of ϕ.

3. The Cubics ϕ and f'. We wish now to put the condition on f such that the triangular relation between it and ϕ will be reciprocal. To do this we shall take ϕ in lines and get another cubic so related to it that triangles will have vertices on it and sides on ϕ; and then identify this new cubic, which we shall call f', with f.

We get the line equation of ϕ by taking a line through two of its points λ_1 and λ_2 and letting the points come together at λ_1, say; that is, by making the λ's equal in the equation

$$x_1 - (\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)x_2 + \lambda_1\lambda_2(\lambda_1 + \lambda_2)x_3 = 0.$$

This gives the tangent at the point λ_1,

$$x_1 - 3\lambda_1^2 x_2 + 2\lambda^3 x_3 = 0,$$

which for varying λ is the line equation of ϕ itself.

We may also get this by the determinant scheme which we shall make use of in what follows, namely,

$$\xi_i = \begin{vmatrix} x_j & x_k \\ x_j' & x_k' \end{vmatrix} \qquad (i, j, k = 1, 2, 3,\ 2, 3, 1,\ 3, 1, 2)$$

where x' is the derivative of x with respect to λ. This holds as well for changing from lines to points as from points to lines by simply interchanging ξ's and x's. This scheme gives, as before, for the parametric form of ϕ in lines

$$\xi_1 : \xi_2 : \xi_3 = 1 : -3\lambda^2 : 2\lambda^3.$$

To get the relation of the parameters of the three points the tangents at which pass through a point we substitute these values in the equation of a point. This gives

$$x_1 - 3\lambda^2 x_2 + 2\lambda^3 x_3 = 0,$$

in which the term in λ is missing. This tells us that the sum of the products of the roots by two's is zero, that is,

$$\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 = 0,$$

or

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0.$$

The triple lines of this involution are given by the roots of

$$\frac{3}{\lambda} = 0,$$

from which we have

$$\lambda = 0,\ 0,\ \infty.$$

The double root 0 gives the flex tangent, that is, the line $x_1 = 0$; and the single root ∞ gives the cusp tangent $x_3 = 0$.

Substituting the parametric values of ξ in the conic $(x\xi)^2 = 0$, and arranging in powers of λ, we get

$$4x_3^2\lambda^6 - 12x_2x_3\lambda^5 + 9x_2^2\lambda^4 + 4x_1x_2\lambda^3 - 6x_1x_2\lambda^2 + x_1^2 = 0,$$

a sextic in λ which gives the six common tangents to the cubic and conic; that is to say, the roots of this equation are the parameters of the points on the cubic ϕ at which these common lines are tangent to the cubic. This equation has no first power term in λ, therefore the sum of the products of its roots by fives vanishes; or

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} + \frac{1}{\lambda_5} + \frac{1}{\lambda_6} = 0.$$

Fixing three of these, say λ_4, λ_5, λ_6, we have as an involution for the other three

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = \kappa_1;$$

that is, if we take tangents at three fixed points of ϕ, and take a conic tangent to these, the points of tangency of the other three common lines are connected by this relation.

Taking three lines on ϕ and one off, analogously to the other case where we had ϕ in points and took three points on and one off, also making use of the involution above, we get as a second relation among the λ's for this case

$$\frac{1}{\lambda_1\lambda_2\lambda_3} - \alpha_1\left(\frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_3\lambda_1}\right) - \alpha_1\beta_1\kappa_1 = 0.$$

By subjecting the point of intersection of two lines of ϕ to these two involutions the combination of which is likewise an I_2^3, we shall get a locus such that triangles with vertices on it will have their sides on ϕ; since it is a locus of triads related to ϕ in such a way that if one point of the triad is given the other two are determined.

The intersection of two lines of ϕ is

$$\begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ 1 & -3\lambda_1^2 & \lambda_1^3 \\ 1 & -3\lambda^2 & \lambda^3 \end{vmatrix} = 0,$$

which reduces to

$$6\lambda_1^2\lambda_2^2\xi_1 + 2(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)\xi_2 + 3(\lambda_1 + \lambda_2)\xi_3 = 0.$$

Subject this point to the conditions expressed in the two involutions; that is, eliminate λ_1 and λ_2 from the three equations and we get, putting λ for λ_3,

$$\begin{aligned}6(\lambda^2 - a_1\lambda^3)\xi_1 - 2\{(a_1\kappa_1^2 + a_1\beta_1\kappa_1)\lambda^3 - (a_1\kappa_1 + \kappa_1^2)\lambda^2 + 2\kappa_1\lambda - 1\}\xi_2 \\ + 3\{a_1\beta_1\kappa_1^2\lambda^3 + (a_1\kappa_1^2 - a_1\beta_1\kappa_1)\lambda^2 - 2a_1\kappa_1\lambda + a_1\}\xi_3 = 0,\end{aligned}$$

which for fixed λ is a point on the curve f', for varying λ the curve itself. Writing it in powers of $1/\lambda$,

$$\begin{aligned}(2\xi_2 + 3a_1\xi_3)\frac{1}{\lambda^3} - 2\kappa_1(2\xi_2 + 3a_1\xi_3)\frac{1}{\lambda^2} + \{6\xi_1 + 2(a_1\kappa_1 + \kappa_1^2)\xi_2 \\ + 3(a_1\kappa_1^2 - a_1\beta_1\kappa_1)\}\frac{1}{\lambda} - \{6a_1\xi_1 + 2(a_1\kappa_1^2 + a_1\beta_1\kappa_1)\xi_2 - 3a_1\beta_1\kappa_1^2\}\xi_3 = 0,\end{aligned}$$

from which we get the symmetrical relation of the λ's

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 2\kappa_1.$$

The condition for triple points in this involution is

$$\frac{3}{\lambda} = 2\kappa_1.$$

The solution of this gives

$$\lambda = 0,\ 0,\ \frac{3}{2\kappa_1},$$

which tells us that this is likewise a cuspidal cubic, the cusp being given for the double root 0, and the flex for the other value of the root $\frac{3}{2\kappa_1}$.

This cubic in parametric form is

$$f':\ \begin{cases} x_1 = 6(\lambda^2 - a_1\lambda^3) \\ x_2 = -2\{a_1\kappa_1^2 + a_1\beta_1\kappa_1)\lambda^3 - (a_1\kappa_1 + \kappa_1^2)\lambda^2 + 2\kappa_1\lambda - 1\} \\ x_3 = 3\{a_1\beta_1\kappa_1^2\lambda^3 + (a_1\kappa_1^2 - a_1\beta_1\kappa_1)\lambda^2 - 2a_1\kappa_1\lambda + a_1\}. \end{cases}$$

4. Identification of f and f'. We wish now to identify f' with f; for, since f is so related to ϕ that triangles with vertices on ϕ have their sides tangent to f, and f' is so related to ϕ that the sides of triangles with vertices on it are tangent to ϕ, by bringing these two, f and f', together we shall have a cubic which has the reciprocal relation with ϕ; or as we say, the two stand in the reciprocal triangular relation or situation.

We have seen that the flex tangent of f goes through the cusp of ϕ. We must put the condition on f, therefore, that will make its cusp lie on the flex tangent of ϕ. This will give us a relation between α, β, and κ.

To get this relation we change f from lines to points; and then, putting in the condition for cusp, namely, $\lambda = \frac{2\kappa}{3}$, we equate the first coordinate x_1 to zero. This gives

$$x_1 \equiv \begin{vmatrix} -2\kappa^3 + 9\alpha\kappa^2 + 27\alpha\beta\kappa, & 2\kappa^3 + 9\beta\kappa^2 \\ \kappa^2 - 3\alpha\kappa, & -\kappa^2 - 3\beta\kappa \end{vmatrix} = 0,$$

which reduces to

$$(1) \qquad (\alpha + \beta)\kappa^2 + 9\alpha\beta\kappa + 27\alpha\beta^2 = 0.$$

As we shall see, f and f' will each have three contacts with ϕ. If these contacts be identified, and also the flex points of the two, the cubics themselves will be completely identified.

We shall first look at the condition for the intersections of the two cubics ϕ and f. Since the points of the triad λ_1, λ_2, λ_3 lie on ϕ and the sides lie on

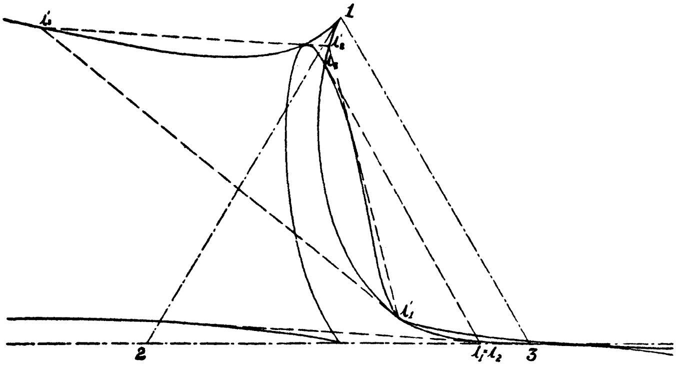

FIG. 1.

f, if we let two of the points, say λ_1 and λ_2, come together, thereby making the lines $\lambda_1\lambda_3$ and $\lambda_2\lambda_3$, which always intersect on ϕ and are tangent to f, coincide, in the limit the points of tangency of the lines with f and the intersection of the lines coincide on ϕ, of course; and the double line becomes tangent to f

at its point of intersection with ϕ. The line $\lambda_1\lambda_2$ has become a tangent to ϕ where this double line cuts ϕ. The involution then becomes

$$2\lambda_1 + \lambda_3 = \kappa;$$

that is, this is the condition for intersections of ϕ and f.

In the case of contacts one of the sides of the triangle, $\lambda_1\lambda_3$ say, becomes a common tangent to the two curves at a vertex, say λ_1.* The two curves have, as we say, a common tangent $\lambda_1\lambda_3$ at this point.† We have, therefore, as the condition for contacts between ϕ and f

$$2\lambda_1 + \lambda_3 = 0.$$

Eliminating λ_2 and λ_3 from the three equations

$$2\lambda_1 + \lambda_3 = 0,$$
$$\lambda_1 + \lambda_2 + \lambda_3 = \kappa,$$
$$\lambda_1\lambda_2\lambda_3 - a(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) - a\beta\kappa = 0,$$

putting λ for λ_1, we get for the equation giving the contacts of ϕ and f

$$(2) \qquad 2\lambda^3 + (2\kappa - 3a)\lambda^2 - a\kappa\lambda + a\beta\kappa = 0,$$

a cubic equation in λ, showing, therefore, that there are three points of contact between the two cubics.

Likewise the contacts of ϕ and f' are given by the condition

$$\frac{2}{\lambda_1} + \frac{1}{\lambda_3} = 0.$$

Eliminating λ_2 and λ_3 from this and the equations of the two involutions

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = \kappa_1,$$

$$\frac{1}{\lambda_1\lambda_2\lambda_3} - a_1\left(\frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_3\lambda_1}\right) - a_1\beta_1\kappa_1 = 0,$$

putting as before λ for λ_1, we get

$$(3) \qquad a_1\beta_1\kappa_1\lambda^3 - a_1\kappa_1\lambda^2 + (2\kappa_1 - 3a_1)\lambda + 2 = 0,$$

the equation giving the three contacts of ϕ and f'.

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* See triangle $\lambda_1'\lambda_2'\lambda_3'$ in figure 1.

† For proof of these for the general case see F. Morley: On Two Cubic Curves in Triangular Relation, *Proceedings of the London Mathematical Society*, ser. 2, vol. 4, part 5.

Comparing equations (2) and (3), we see that in order for these to be the same, that is, in order for f and f' to have the same contacts with ϕ, we have the relation among the coefficients

$$(4)\qquad \frac{2}{\alpha_1\beta_1\kappa_1} = -\frac{2\kappa - 3\alpha}{\alpha_1\kappa_1} = -\frac{\alpha\kappa}{2\kappa_1 - 3\alpha_1} = \frac{\alpha\beta\kappa}{2},$$

from which we infer that the constants with subscripts may be interchanged with those without them.

We shall next identify the flex points of f and f'. We get the flex point of f by writing it in points and putting in the condition for a flex, namely, $\lambda = \infty$. By the determinant scheme the coordinates of any point x on f are seen to be

$$x_1 = -\alpha \begin{vmatrix} \lambda(\lambda - \kappa)^2 + \alpha\kappa\,(\lambda - \beta - \kappa), & \lambda(\lambda - \kappa)^2 - \beta\kappa(\lambda - \kappa) \\ (\lambda - \kappa)^2 + 2\lambda(\lambda - \kappa) + \alpha\kappa, & (\lambda - \kappa)^2 + 2\lambda(\lambda - \kappa) - \beta\kappa \end{vmatrix}$$

$$x_2 = \alpha \begin{vmatrix} \lambda(\lambda - \kappa)^2 - \beta\kappa(\lambda - \kappa), & \lambda - \alpha \\ (\lambda - \kappa)^2 + 2\lambda(\lambda - \kappa) - \beta\kappa, & 1 \end{vmatrix}$$

$$x_3 = -\begin{vmatrix} \lambda - \alpha, & \lambda(\lambda - \kappa)^2 + \alpha\kappa(\lambda - \beta - \kappa) \\ 1, & (\lambda - \kappa)^2 + 2\lambda(\lambda - \kappa) + \alpha\kappa \end{vmatrix}.$$

Putting $\lambda = \infty$ in these, we find that they reduce, respectively, to

$$\alpha\kappa(\alpha + \beta),\ \alpha,\ 1,$$

which are the coordinates of the flex point of f. Or we have as the equation giving this flex point

$$(5)\qquad \alpha\kappa(\alpha + \beta)\xi_1 + \alpha\xi_2 + \xi_3 = 0.$$

The flex point of f' is obtained by putting $\lambda = \frac{3}{2\kappa_1}$ in the equation of f'. When this value of λ is put in, the equation of f' reduces to

$$(6)\quad 54(2\kappa_1 - 3\alpha_1)\xi_1 - 2\kappa_1(9\alpha_1\kappa_1 - 2\kappa_1^2 + 27\alpha_1\beta_1)\xi_2 + 3\kappa_1(9\alpha_1\beta_1\kappa_1 + 2\alpha_1\kappa_1^2)\xi_3 = 0.$$

If now these two flex points are to be the same, we must have the relation

$$(7)\quad \frac{54(2\kappa_1 - 3\alpha_1)}{\alpha\kappa(\alpha + \beta)} = -\frac{2\kappa_1(9\alpha_1\kappa_1 - 2\kappa_1^2 + 27\alpha_1\beta_1)}{\alpha} = \frac{3\kappa_1(9\alpha_1\beta_1\kappa_1 + 2\alpha_1\kappa_1^2)}{1}.$$

Hence, this is the condition that the flexes of f and f' coincide.

The conditions expressed in (4) and (7) are sufficient to identify the two cubics completely; for, to say they shall have contact at the same point is two conditions, therefore six conditions for the three contacts; and identifying the flex points is one other condition, making in all seven. But seven conditions completely fix a cuspidal cubic, since requiring it to have a cusp is equivalent to two conditions.

5. Equation of f in terms of β. We have already found one relation connecting α, β, and κ, namely, equation (1). From (4) and (7) we shall find another, and by means of these two shall be able to express two of the constants in terms of the third.

To avoid writing subscripts interchange the constants α_1, β_1, κ_1 with α, β, κ, which we saw by (4) could be done, in expressions (4) and (7), and eliminate α_1, β_1, κ_1 from the resulting expressions, namely,

$$(4') \qquad \frac{2}{\alpha\beta\kappa} = -\frac{2\kappa_1 - 3\alpha_1}{\alpha\kappa} = -\frac{\alpha_1\kappa_1}{2\kappa - 3\alpha} = \frac{\alpha_1\beta_1\kappa_1}{2},$$

$$(7') \qquad \frac{54(2\kappa - 3\alpha)}{\alpha_1\kappa_1(\alpha_1 + \beta_1)} = -\frac{2\kappa(9\alpha\kappa - 2\kappa^2 + 27\alpha\beta)}{\alpha_1} = 3\kappa^2(9\alpha\beta\kappa + 2\alpha\kappa^2).$$

This elimination gives

$$(8) \qquad 8\kappa^3 - 60\alpha\kappa^2 + 54\alpha^2\kappa - 108\alpha\beta\kappa + 81\alpha^2\beta = 0.$$

And now eliminating α from (1) and (8), we get

$$(9) \qquad 8\kappa^4 + 204\beta\kappa^3 + 1782\beta^2\kappa^2 + 6561\beta^3\kappa + 8748\beta^4 = 0,$$

a homogeneous equation of the fourth degree in β and κ.

The solution of this equation gives $\frac{\kappa}{\beta} = -\frac{9}{2}$ or -12, and by putting these in (1) we find also $\frac{\alpha}{\beta} = -3$ or $-\frac{16}{7}$. Taking the last set of these, namely

$$\alpha = -\frac{16}{7}\beta, \quad \kappa = -12\beta,$$

and substituting in the original equation of f, we get it in the form

$$f: \begin{cases} \xi_1 = \lambda + \frac{16}{7}\beta \\ \xi_2 = -\lambda^3 - 24\beta\lambda^2 - \frac{1200}{7}\beta^2\lambda - \frac{2112}{7}\beta^3 \\ \xi_3 = -\frac{16}{7}\beta\lambda^3 - \frac{384}{7}\beta\lambda^2 - \frac{2496}{7}\beta^3\lambda - \frac{2304}{7}\beta^4. \end{cases}$$

6. Constants and Conditions. We have found that a and κ may be expressed in terms of β, which may be given an infinite number of values; and each of these gives a cubic in the desired relation. There is therefore a single infinity of cubics satisfying the requirements, namely, having this reciprocal triangular relation with the original cubic.

This follows also from consideration of the conditions imposed. To say a cubic shall have a cusp imposes two conditions upon it, leaving seven conditions, or seven degrees of freedom for the cuspidal cubic. To say there shall be three contacts is equivalent to three conditions, and to make this reciprocal puts on three more. There is left then one degree of freedom, which says likewise that there is a single infinity of cubics in the proper situation.

7. Contacts of ϕ and f. The equation giving the contacts of ϕ and f we have obtained as

$$2\lambda^3 + (2\kappa - 3a)\lambda^2 - a\kappa\lambda + a\beta\kappa = 0.$$

On substitution for κ and a in terms of β this becomes

$$7\lambda^3 - 60\beta\lambda^2 - 96\beta^2\lambda + 96\beta^3 = 0.$$

The discriminant of this equation is of the form

$$D = A\beta^6,$$

where A is a positive number, and is positive for all values of β. The equation, therefore, has three real roots whatever value we give to β; that is, the curves always have three real contacts.

For β positive there are two positive roots and one negative, that is, two contacts with the part of ϕ which lies within the triangle of reference, and one for the part outside. For β negative there is one contact within and two without the triangle; since one root is positive and two are negative.

We shall see that $\beta = \pm n$ gives the cusps of f on x_1 harmonic with the reference points 0, 1, 0 and 0, 0, 1. So here, the contacts given by $\beta = \pm n$ lie on a line through 0, 0, 1 and are harmonic with this point and the point where x_3 cuts the line; that is to say, lines through 0, 0, 1 and the points of contact for $\beta = n$ cut ϕ again at the points of contact for $\beta = -n$.

8. Intersections of ϕ and f. To get the equation giving the intersections we eliminate λ_1 and λ_2 from

$$2\lambda_1^3 + \lambda_3 = \kappa,$$

which is the condition for intersections, where λ_3 is the parameter of this point, and the involutions

$$\lambda_1 + \lambda_2 + \lambda_3 = \kappa,$$

$$\lambda_1\lambda_2\lambda_3 - a(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) - a\beta\kappa = 0.$$

Calling λ_3, λ, this gives

$$\lambda^3 + (3a - 2\kappa)\lambda^2 + (\kappa^2 - 2a\kappa)\lambda - a\kappa^2 - 4a\beta\kappa = 0.$$

The cubics, therefore, have three intersections.

Making the substitution $a = -\frac{16}{7}\beta$, $\kappa = -12\beta$ in this equation, we get

$$7\lambda^3 + 120\beta\lambda^2 + 624\beta^2\lambda + 1536\beta^3 = 0.$$

The discriminant of this is of the form

$$D = -A'\beta^6;$$

that is, it is negative whatever value β may have, which tells us there is only one real root. The cubics, therefore, have but one real intersection.

9. Characteristics of the System. As different values are given to β in the equation of f, the cusp of f takes different positions along the line x_1. As β varies from 0 to $-\infty$ the cusp passes from the point 0, 0, 1 to 0, 1, 0, the cusp tangent passing from coincidence with x_1 to coincidence with x_3, and the flex tangent at the same time passing from coincidence with x_2 to coincidence with x_3. While β is taking all these negative values, there is one contact in the triangle and two outside; and the real intersection is in the triangle.

For positive values of β the cusp traverses the external segment of the side x_1, passing from the point 0, 1, 0 around to 0, 0, 1 through infinity as β diminishes from $+\infty$ to 0. Here there are two contacts in the triangle and one outside. Also, in this case the intersection of ϕ and f is outside.

The cusp of f is at the middle point of the base for $\beta = -\frac{1}{4}$. For $\beta = \frac{1}{4}$ it is at infinity, the cusp tangent being parallel to the base for this value. In general $\beta = \pm n$ gives two positions of the cusp, that is, two points on x_1 which are harmonic with 0, 1, 0 and 0, 0, 1.

10. Degenerate cases of f. For certain values of β the cubic f will degenerate, and we wish to consider these degenerate cases.

When the cusp tangent is x_1 and the flex tangent is x_2; that is, when $\beta = 0$, the cusp and flex coincide at their intersection 0, 0, 1. The cubic, therefore, must break up for this value of β. To show how it breaks up substitute $d\beta$ for β in the equation of f. Disregarding higher powers of $d\beta$ than the first, we write f simply

$$\xi_1 = \lambda + \frac{16}{7}d\beta, \quad \xi_2 = -\lambda^3 - 24d\beta\lambda^2, \quad \xi_3 = -\frac{16}{7}d\beta\lambda^3;$$

or let us say

$$\xi_1 = \lambda + a, \quad \xi_2 = -\lambda^3 - b\lambda^2, \quad \xi_3 = -c\lambda^3,$$

where a, b, c are multiples of $d\beta$. Building up a vanishing function in these infinitesimals, we get the equation

$$ac^2\xi_2^3 - \frac{a+b}{c}\xi_3^3 - b^3\xi_1\xi_2^2 - (3ac+bc)\xi_2^2\xi_3 + (3a+2b)\xi_2\xi_3^2 = 0.$$

Now making $d\beta$ vanish, since the coefficient of ξ_3^3 is the only one independent of $d\beta$, $d\beta$ dividing out and leaving only a numerical quantity, we have simply

$$\xi_3^3 = 0,$$

which is the point 0, 0, 1 three times over.

Likewise, when $\beta = \infty$, the cusp tangent coincides with the flex tangent in x_3. The cusp falls at 0, 1, 0 and the flex at 1, 0, 0; and the cubic again breaks up.

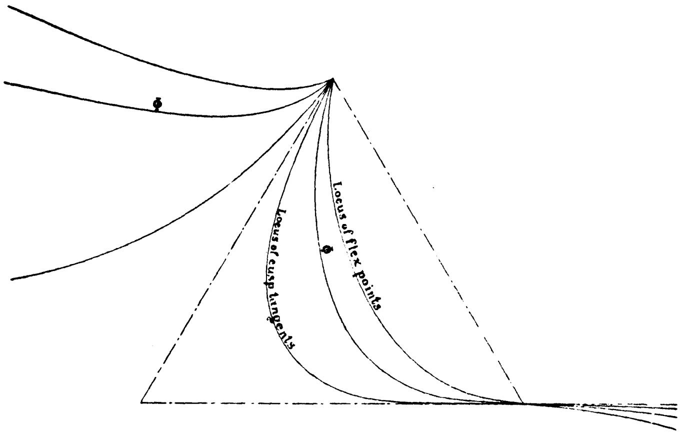

FIG. 2.

11. Locus of the Cusp Tangent and Flex. If the condition for cusp tangent, $\lambda = \frac{2}{3}\kappa = -8\beta$, is put in f, this reduces to the cusp tangent, and gives as its equation

$$\xi_1 = -1, \quad \xi_2 = 8\beta^2, \quad \xi_3 = 32\beta^3.$$

This for a given β is the cusp tangent of the particular cubic of the system given by that β. For varying β it is the locus of the cusp tangents of the whole system.

From its form we recognize this as a cubic which has a cusp for the double root $\beta = 0$ and a flex for $\beta = \infty$; that is, it has its cusp and flex at 1, 0, 0 and 0, 0, 1, respectively, the same as the original cubic.

The flex point of f we have found was

$$x_1 = a\kappa(a + \beta), \quad x_2 = a, \quad x_3 = 1,$$

which in terms of β is

$$x_1 = -1728\beta^3, \quad x_2 = -112\beta, \quad x_3 = 49.$$

Its locus for varying β is likewise another cubic with cusp and flex coincident with those of ϕ.

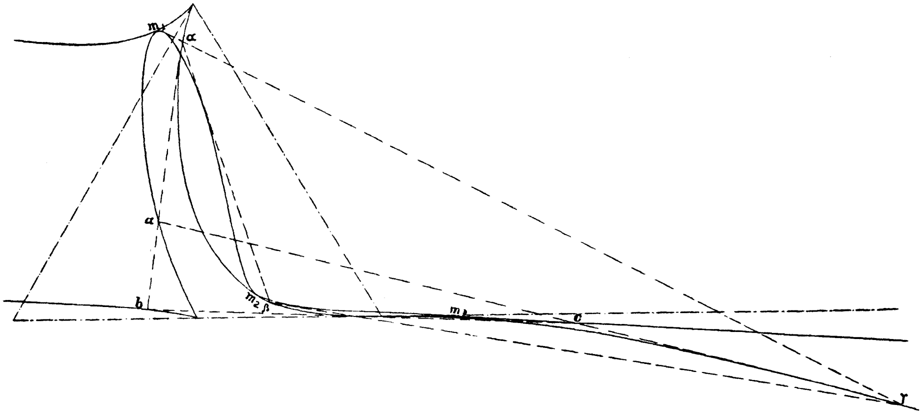

FIG. 3.

12. The Figure. In figure 3 the cubic f is drawn for the case $\beta = -\frac{1}{4}$; its cusp, therefore, lies at the middle point of x_1. The cusp tangent

$$x_1 - 8\beta x_2 - 32\beta^3 x_3 = 0$$

becomes

$$2x_1 - x_2 + x_3 = 0$$

and cuts x_3, therefore, at one third the distance from 0, 1, 0 to 1, 0, 0.

The flex tangent

$$x_2 + \frac{16}{7}\beta x_3 = 0$$

becomes

$$7x_2 - 4x_3 = 0$$

and passes through 1, 0, 0 and cuts x_1 in the ratio 7 : 4. The flex point

$$1728\beta^3\xi_1 + 112\beta\xi_2 - 49\xi_3 = 0$$

becomes

$$27\xi_1 + 28\xi_2 + 49\xi_3 = 0.$$

The coordinates of the point i, therefore, are 27, 28, 49.

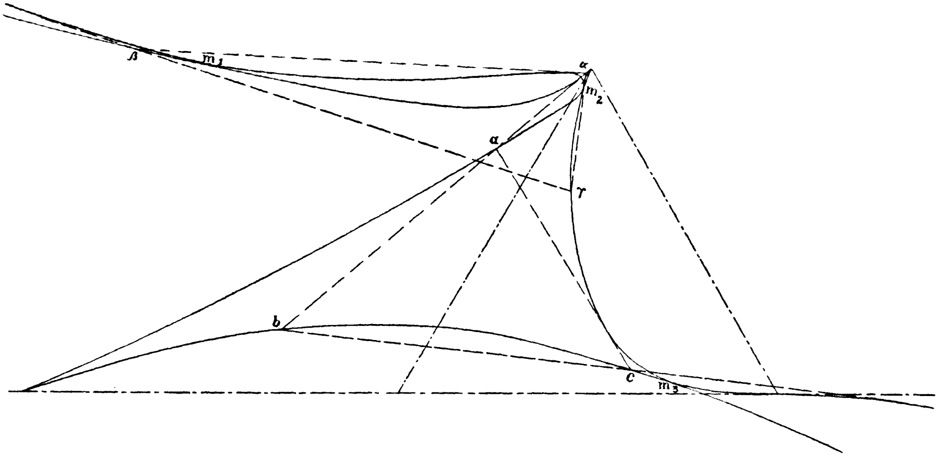

FIG. 4.

The contacts m_1, m_2, m_3 are given by the roots of the equation

$$14\lambda^3 + 30\lambda^2 - 12\lambda - 3 = 0,$$

and are approximately -2.46, $.51$, $-.18$. The intersection is given by the real root of

$$7\lambda^3 - 30\lambda^2 + 39\lambda - 24 = 0,$$

and lies between 2 and 3.

In figure 4 cubic f is drawn for $\beta = \frac{1}{2}$. Its cusp tangent is

$$x_1 - 2x_2 - 4x_3 = 0.$$

The cusp, given by $x_2 + 2x_3 = 0$, lies to the left of 0, 1, 0, and at a distance equal to the base of the reference triangle from it.

The flex tangent

$$7x_2 + 8x_3 = 0$$

also cuts x_1 to the left of 0, 1, 0 in the ratio $7 : -8$. The flex point is

$$216\xi_1 + 56\xi_2 + 49\xi_3 = 0.$$

The contacts m_1, m_2, m_3 are given by the equation

$$7\lambda^3 - 30\lambda^2 - 24\lambda + 12 = 0,$$

the roots of which corresponding to these points, respectively, are approximately $-.99$, 4.9, $.35$.

The equation for the intersections for this case is

$$7\lambda^3 + 60\lambda^2 + 156\lambda + 192 = 0,$$

the real root of which is approximately -5. The intersection, therefore, as we have already noticed, is outside of the triangle.

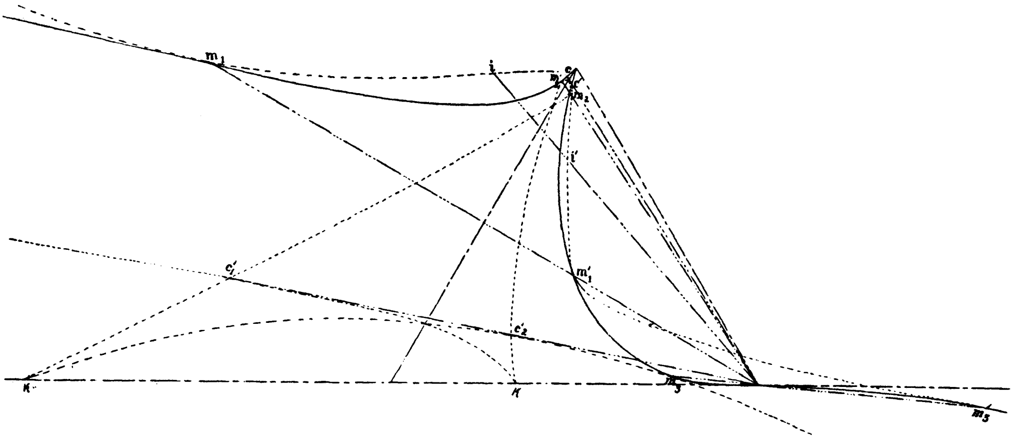

FIG. 5.

In each of the figures 3 and 4 $\alpha\beta\gamma$ is a triangle with vertices on ϕ and sides on f; and abc is one with vertices on f and sides on ϕ.

In figure 5 are shown the two positions of f for $\beta = \frac{1}{2}$ and $\beta = -\frac{1}{2}$. It shows the relation of the contacts of these with ϕ. It will be noticed that the contacts for $\beta = \frac{1}{2}$, say m_1, m_2, m_3, lie on lines with m_1', m_2', m_3', respectively, through 0, 0, 1, m_1', m_2', m_3' being the contacts for $\beta = -\frac{1}{2}$. We have already noticed that these pairs are in an harmonic ratio with the point 0, 0, 1 and the point where x_3 cuts their join. This holds for the cusps κ and κ', the flexes i and i', the points of intersection c and c', and also the two points of intersection of the cubics themselves c_1' and c_2'. The other three points of intersection lie on x_3. In fact the curves are harmonic with 0, 0, 1 and x_3.

There is a definite relation between the cusp tangent and flex tangent for any position of f, or what is the same thing, the cusp and the point where the flex tangent cuts x_1. From the cusp tangent

$$x_1 - 8\beta^2 x_2 - 32\beta^3 x_3 = 0$$

we see that the cusp is given by

$$x_2 + 4\beta x_3 = 0.$$

Where the flex line cuts is given by

$$x_2' + \frac{16}{7}\beta x_3' = 0,$$

showing the one to one correspondence between the two. From these two equations we get the invariant relation or ratio

$$\frac{x_2}{x_3} = \frac{7}{4}\frac{x_2'}{x_3'} \qquad \text{or} \qquad \frac{x_2}{x_3} : \frac{x_2'}{x_3'} = \frac{7}{4}.$$

Starting, say, at the point 0, 0, 1 when $\beta = 0$, for which value the cusp lies at this point, and the flex tangent which is x_2 cuts x_1 at this point also, the cusp and the point of intersection of the flex tangent with x_1 moves to the left for negative values of β, the cusp at first having the greater velocity, let us say. The relative velocities change, and the cusp is overtaken and passed by the other point at 0, 1, 0 for $\beta = \infty$. They then traverse the external segment of the base for positive values of β, the relative velocities being interchanged with the above, the cusp overtaking the other point when 0, 0, 1 is reached, the ratio always being

$$\frac{x_2}{x_3} : \frac{x_2'}{x_3'} = \frac{7}{4}.$$

JOHNS HOPKINS UNIVERSITY,
BALTIMORE, MD.
APRIL, 1906.